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QCD_2 -Screening, Confinement and Novel non-abelian Solutions

Y. FRISHMAN^{*}*Department of Particle Physics**Weizmann Institute of Science**76100 Rehovot Israel*J. SONNENSCHNEIN[†]*School of Physics and Astronomy**Beverly and Raymond Sackler**Faculty of Exact Sciences**Ramat Aviv Tel-Aviv, 69987, Israel*

ABSTRACT

We analyze the question of screening versus confinement in bosonized massless QCD in two dimensions. We deduce the screening behavior of massless $SU(N_c)$ QCD with flavored fundamental fermions and fermions in the adjoint representation. This is done by computing the potential between external quarks as well as by bosonizing also the external sources and analyzing the states of the combined system. We write down novel “non-abelian Schwinger like” solutions of the equations of motion, compute their masses and argue that an exchange of massive modes of this type is associated with the screening mechanism.

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1. INTRODUCTION

The question of confinement versus screening in four dimensional (4D) non-abelian gauge theories is one of the major problems of high-energy theory. Two dimensional (2D) gauge theories may serve as a laboratory in exploring that problem. Just as in 4D, also in 2D one can use the potential between two heavy external charges, the expectation value of a Wilson loop and the structure of the bound state spectrum as probes of confinement. It is believed that a potential growing linearly at large separation distances, an area law behaviour of the Wilson loop and a spectrum which is independent of the number of colors N_c indicate that the system is in a confining state. Note that in some cases, like with Higgs in the fundamental, screening and confinement are in one phase by “complementarity”^[1]

In a paper by D.Gross et al.^[2] it was argued that there is a screening effect between heavy external charges induced by massless dynamical fermions even if the latter are in a representation which has zero “ N_c -ality”, namely, vanishing center ($Z_{N_c} \sim 0$). It was further shown that confinement is restored as soon as the dynamical fermions get some non-trivial mass. In that paper both the nature of the potential and the Wilson loop were determined in the abelian theory and in several non-abelian cases. In one case the group was $SU(N_c = 2)$ with the dynamical fermions in **3**, and the other case was $SU(3)$ with **8** fermions. In the latter it was shown in fact that the spinor **8** of $SO(8)$ are screening.

The potential between the external quarks can be extracted using several different methods: (i) Deriving the effective Lagrangian (integrating out the fermions) and then extracting the potential using the static gauge configurations that solve the corresponding equations of motion.^[2] (ii) Using the gauge configuration that solves the equations of motion of the bosonized action^[3]. (iii) Eliminating the gauge fields from the bosonized gauged action, solving for static solutions of the currents and deducing the potential as the difference between the Hamiltonian of the systems with and without the external sources.

In the present paper we use method (ii) to prove the statements of screening

for massless dynamical fermions.

Another approach to determine if the system confines is based on bosonizing also the heavy external charges.^[4] Confinement manifests itself in this double bosonized model by the absence of soliton solutions that correspond to unbounded quarks. In case that there are quark finite-energy static solutions, one may conclude that the system is non-confining. We use also this type of analysis for both abelian and non-abelian gauge theories.

The screening mechanism in the massless Schwinger model could be attributed to the exchange of the emerged massive photon, which is the only state in the exact spectrum. The non-abelian counterpart is clearly much more complicated and seems to be a non-integrable model^[5]. However, by introducing flavor degrees of freedom one can pass, in the limit of large number of flavours N_f (with finite N_c), to a domain where the non abelian theory resembles a collection of $N_c^2 - 1$ abelian theories. In that limit the spectrum includes $N_c^2 - 1$ massive modes of the type that exist in the Schwinger model. One can then draw an intuitive picture of screening due to those modes in a similar manner to the one in the abelian theory. As a matter of fact, it is only in the large N_f limit, that one can justify relating solutions of the equations of motion and physical states and deducing conclusions about the structure of the spectrum. One might find in the “massive gauge states” an indication of the “non-confining” structure of the spectrum. The reason for that is, that even though they are gauge invariant states, they are in the adjoint representation of a “global color symmetry” and not singlets of that group. These states had already been pointed out in an earlier work,^[6] based on a BRST analysis and a special parametrization of the gauge configurations. However, in that paper we were not able to rigorously show that indeed they were part of the BRST cohomology. Note, that even if they are not in space of physical states, the massive states could nevertheless be responsible for the screening potential.

When passing from a screening picture at large N_f and finite N_c to the domain of a small number of flavors, one can anticipate two types of scenarios: (i) A smooth

transition where the screening behavior persists all the way down to $N_f = 1$; (ii) A phase transition at a certain value of N_f and a confining nature below it. One may argue that the massive modes of large N_f are an artifact of the abelianization of the theory. To check that possibility we have searched for non-abelian solutions of the equations of motions. Indeed, we found new non-abelian solutions that are also massive and are associated with the gauge fields, namely, are in the adjoint of the global color group. We thus conclude that this nature does not stem from the abelianization of the large N_f limit, and hence may present certain evidence in favour of option (i). A different “patch” of the space of physical states for finite N_c and finite N_f , that of the low lying baryonic states, was determined in the semiclassical domain in [7]. Since those baryons are all color singlets this might seem as a contradiction to the screening nature of the spectrum. In fact there is no contradiction since the baryons were discovered only for massive quarks and not for massless ones. As was shown in [2] turning on a mass term for the quarks changes the picture dramatically into a confining one.

't Hooft solved the spectrum of QCD_2 in the large N_c limit.^[8] In that analysis the quarks were flavorless and in the color fundamental representation. This procedure was recently also applied for adjoint fermions. Both in the original work as well as in those of ref. [9] there is no trace of the massive modes that our work analyzes. Differently stated the large N_c approach reveals a confining spectrum both for the case of massless and massive quarks. We believe that this is an artifact of the large N_c limit and at finite N_c the spectrum of a theory with/without quark mass behaves like a confining/screening spectrum respectively.

The paper is organized in the following way. In section 2 we review the rules of bosonization of two dimensional QCD with both massive and massless fermions which transform in the fundamental or adjoint representations. The equations of motions of bosonized QCD_2 in the presence of external currents are derived and discussed in section 3. Non-abelian solutions of the equations as well as some interesting abelian ones are presented in section 4 for the model without external sources, and in section 5 when the latter are turned on. The energy-momentum

tensor and the spectrum that corresponds to the solutions of sections 4 and 5 are derived in section 6. Section 7 is devoted to the analysis of the system by bosonizing also the external currents. In section 8 we summarize the results of the present work, state our understanding of the nature of QCD_2 in the different regimes and raise some further interesting open questions. In appendix A we find some “truly non-abelian” $SU(N_c = 2)$ solutions of the equations of motion of bosonized QCD_2 . Appendix B is devoted to the derivation of the energy momentum tensor of a bosonized fermion coupled to an abelian gauge field and its non-abelian generalization. In appendix C we show that minimal energy solutions of the massive WZW model are necessarily in a diagonal form.

2. REVIEW OF BOSONIZATION IN QCD_2

We start with reviewing the bosonization formulations of QCD_2 with fermions in the fundamental and adjoint representations.

Dirac fermions in the fundamental representation.

Multiflavor massive QCD_2 with fermions in the fundamental representation was shown^[7] to be described by the following action

$$S_{QCD_2} = S_1(u) - \frac{1}{2\pi} \int d^2z \text{Tr}(iu^{-1}\partial u \bar{A} + iu\bar{\partial}u^{-1}A + \bar{A}u^{-1}Au - A\bar{A}) + \frac{m^2}{2\pi} \int d^2z : \text{Tr}_G[u + u^{-1}] : + \frac{1}{e_c^2} \int d^2z \text{Tr}_H[F^2] \quad (1)$$

where $u \in U(N_f \times N_C)$; $S_k(u)$ is a level k WZW model;

$$S_k(u) = \frac{k}{8\pi} \int d^2x \text{Tr}(\partial_\mu u \partial^\mu u^{-1}) + \frac{k}{12\pi} \int_B d^3y \varepsilon^{ijk} \text{Tr}(u^{-1}\partial_i u)(u^{-1}\partial_j u)(u^{-1}\partial_k u) \quad (2)$$

A and \bar{A} take their values in the algebra of $H \equiv SU(N_C)$; $F = \bar{\partial}A - \partial\bar{A} + i[A, \bar{A}]$; m^2 equals $m_q\mu C$, where μ is the normal ordering mass and $C = \frac{1}{2}e^\gamma$ with γ Euler’s constant. Notice that the space-time has a Minkowski signature, and we

use a notation in which the light-cone components of a vector B_μ are denoted by $B \equiv B_+$ and $\bar{B} \equiv B_-$.

The action for massless fermions can be simplified using the following parametrization $u \equiv ghle^{i\sqrt{\frac{4\pi}{N_c N_f}}\phi}$ where $h \in SU(N_c)$, $g \in SU(N_f)$ and $l \in \frac{U(N_c N_f)}{[SU(N_c)]_{N_f} \times [SU(N_f)]_{N_c} \times U_B(1)}$. The action in terms of these variable takes the form^[3]

$$\begin{aligned} S = & S_{N_f}(h) - \frac{N_f}{2\pi} \int d^2z \text{Tr}(ih^{-1}\partial h\bar{A} + ih\bar{\partial}h^{-1}A + \bar{A}h^{-1}Ah - A\bar{A}) \\ & + S_{N_c}(g) + \frac{1}{2\pi} \int d^2z [\partial\phi\bar{\partial}\phi] \\ & + \frac{1}{e_c^2} \int d^2z \text{Tr}_H[F^2] \end{aligned} \quad (3)$$

Notice that the action is independent of l .^[3] Recall that to discuss the quark soliton structure we need the massive $u \in U(N_f \times N_c)$ description.^[4]

Majorana fermions in the adjoint representation.

A non-abelian bosonization of Majorana fermions in the adjoint representation^[10] can be expressed in terms of $S(h_{ad})$ where h_{ad} are $(N_c^2 - 1) \times (N_c^2 - 1)$ matrices, so that the action for the corresponding QCD_2 now reads, in the massive case,

$$\begin{aligned} S = & \frac{1}{2}S(h_{ad}) - \frac{1}{4\pi} \int d^2z \text{Tr}(ih_{ad}^{-1}\partial h_{ad}\bar{A} + iu\bar{\partial}h_{ad}^{-1}A + \bar{A}h_{ad}^{-1}Ah_{ad} - A\bar{A}) \\ & + \frac{m^2}{2\pi} \int d^2z \text{Tr}_G : [h_{ad} + h_{ad}^{-1}] : \\ & + \frac{1}{e_c^2} \int d^2z \text{Tr}_H[F^2] \end{aligned} \quad (4)$$

The factor $\frac{1}{2}$ in front of the $S(h_{ad})$ term comes from the reality nature of the Majorana fermions. It is straightforward to realize that the conformal anomaly of this model is indeed $c = \frac{1}{2}(N_c^2 - 1)$, the associated currents have affine Lie algebra with an anomaly of N_c and that the conformal dimension of h_{ad} is $\Delta h_{ad} = \frac{1}{2}$ (left and right dimensions and a total conformal dimension of 1).

3. EQUATIONS OF MOTION OF QCD_2 IN THE PRESENCE OF EXTERNAL CURRENTS

The equation of motions which follow from the variation of the action (3) with respect to h are given for the massless case by

$$\begin{aligned}\bar{\partial}(h^{-1}\partial h) + i\partial\bar{A} + i[h^{-1}\partial h, \bar{A}] &= 0 \\ \partial(h\bar{\partial}h^{-1}) - i\partial(h\bar{A}h^{-1}) &= 0\end{aligned}\tag{5}$$

where a gauge $A = 0$ has been chosen. Notice that the second equation can be derived from the first one by multiplying it with h from the left and h^{-1} from the right. A similar result but with h_{ad} replacing h follows the variation of eqn. (4) with respect to h_{ad} .

As was discussed in the introduction our aim is to analyze the system in the presence of external sources. External currents are coupled to the system by adding to the action (3) or (4) the following term

$$\mathcal{L}_{ext} = \frac{1}{2\pi} \int d^2z \text{Tr}(J_{ext}\bar{A} + \bar{J}_{ext}A).$$

The variation of the combined action with respect to A and \bar{A} (and then setting $A = 0$) yields for the case of N_f fundamentals the following equations of motion

$$\begin{aligned}\partial^2\bar{A} + \alpha_c(iN_f h^{-1}\partial h + J_{ext}) &= 0 \\ \partial\bar{\partial}\bar{A} + [i\partial\bar{A}, \bar{A}] - \alpha_c[N_f(ih\bar{\partial}h^{-1} + h\bar{A}h^{-1} - \bar{A}) + \bar{J}_{ext}] &= 0\end{aligned}\tag{6}$$

where $\alpha_c = \frac{e_c^2}{4\pi}$. It follows from the equations of motion (5) and (6) that both the dynamical currents $j_{dy} = \frac{iN_f}{2\pi}h^{-1}\partial h$, $\bar{j}_{dy} = \frac{iN_f}{2\pi}[h\bar{\partial}h^{-1} - ih\bar{A}h^{-1} + i\bar{A}]$ as well as the external currents are covariantly conserved, which for $A = 0$ reads

$$\bar{D}j_{dy} + \partial\bar{j}_{dy} = 0 \quad \bar{D}J_{ext} + \partial\bar{J}_{ext} = 0.\tag{7}$$

with $\bar{D} = \bar{\partial} - i[\bar{A}, \cdot]$. One can eliminate the dynamical current and derive the

following equation for the gauge fields in terms of the external currents

$$\partial\bar{\partial}\bar{A} + [i\partial\bar{A}, \bar{A}] + \alpha_c(N_f\bar{A} - \bar{J}_{ext}) = 0 \quad (8)$$

In fact the equation one gets in this way is the ∂ derivative acting on the l.h.s of (8) equals zero. However, one can fix the residual gauge invariance $\bar{A} \rightarrow iu^{-1}\bar{\partial}u + u^{-1}\bar{A}u$, with u an anti-holomorphic function $\partial u = 0$ (thus preserving $A = 0$), to eliminate the antiholomorphic function that should have been put in the r.h.s of (8). Thus $\bar{A} = ih^{-1}\bar{\partial}h$ and $\bar{j}_{dy} = -\frac{N_f}{2\pi}\bar{A} = -\frac{iN_f}{2\pi}h^{-1}\bar{\partial}h$. Note that this is not the current of the free case which is $\bar{j}_{free} = \frac{iN_f}{2\pi}h\bar{\partial}h^{-1} = -\frac{iN_f}{2\pi}(\bar{\partial}h)h^{-1}$

The last equation, which holds for the massless cases, (3) and $m = 0$ of (4), is universal in the sense that it is independent of the representation of the dynamical fermions^[11]. Once mass terms are added there is an explicit dependence on the dynamical fermion and instead of (8) one finds that the ∂ derivative of its left hand side is equal to $im^2N_f\alpha_c(h - h^{-1})$, where m^2 is given in eqn. (2) and where for the fundamental representation we have chosen non-flavored configurations namely we set g, l, ϕ of eqn.(3) to 1, 1, 0 respectively (otherwise instead of a factor N_f , h is multiplied by the flavored contribution $Trgle^{i\sqrt{\frac{4\pi}{N_C N_f}}\phi}$).

Studying the quantum system by analyzing the corresponding equations of motion is a justified approximation only provided that the classical configurations dominate the functional integral. Such a scenario can be achieved for the case of massless quarks in the fundamental color representation in the limit of a large number of flavours. In fact one can show that in that case $\frac{1}{N_f}$ plays the role of Λ . The colored sectors of the action takes the form

$$S = N_f \left\{ S_1(h) - \frac{1}{2\pi} \int d^2z Tr(ih^{-1}\partial h\bar{A}) + \frac{1}{\tilde{e}_c^2} \int d^2z Tr[\partial\bar{A}]^2 \right\} \quad (9)$$

where $\tilde{e}_c = e_c\sqrt{N_f}$.

4. SOLUTIONS OF THE EQUATIONS WITHOUT EXTERNAL QUARKS

Let us consider first the case where the external sources are switched off. It is obvious that an “abelian” massive mode is a solution of the equations of motion (8). Consider a configuration of the form $\bar{A} \equiv T^a \bar{A}^a(z, \bar{z}) = T^a \delta^{a, a_0} \bar{\mathcal{A}}(z, \bar{z})$, where a_0 is a given index that takes one of the values $1, \dots, N_c^2 - 1$, then the commutator term vanishes and $\bar{\mathcal{A}}$ has to solve $\partial \bar{\partial} \bar{\mathcal{A}} + \tilde{\alpha}_c \bar{\mathcal{A}} = 0$ with $\tilde{\alpha}_c = N_f \alpha_c$. It is clear that there are $N_c^2 - 1$ such solutions and in fact it is easy to see that this property will be shared by every possible solution. This follows from the fact that the equation of motion is not invariant but rather covariant with respect to the “global color” transformation $A \rightarrow u^{-1} A u$ with a constant u .

Let us now check whether the equations admit soliton solutions. For static configurations the equation reads

$$\partial_1^2 \bar{A} - \sqrt{2} [i \partial_1 \bar{A}, \bar{A}] - 2 \tilde{\alpha}_c \bar{A} = 0 \quad (10)$$

Multiplying the equation by \bar{A} , taking the trace of the result and integrating over dx one finds after a partial integration that $\int dx [Tr[(\partial \bar{A})^2 + 2 \tilde{\alpha}_c \bar{A}^2] = 0$ which can be satisfied only for a vanishing \bar{A} .

In the search for other possible solutions one may be instructed by the fact that $F_{z\bar{z}} = \partial \bar{A}$ can be written in 2d as $F_{z\bar{z}} = \epsilon_{z\bar{z}} F$ and impose an ansatz for the solution of the form $\bar{A} = z C(\rho)$ where $\rho = z\bar{z}$. Expanding C as a power series in ρ one finds that the commutator term has to vanish and C is determined by the equation $\rho C''' + 2C' + \tilde{\alpha}_c C = 0$, where $C' = \partial_\rho C$. By a change of variable $C = \rho^{\frac{1}{2}} W(x)$ one can rewrite this equation as $W'' + \frac{1}{x} W' + (1 - \frac{1}{x^2}) W = 0$ with $x^2 = 4 \tilde{\alpha}_c \rho$, which is x^{-2} times a Bessel equation of order one, so that the solution for the gauge field takes the form

$$\bar{A} = \frac{\bar{A}_0 z}{\sqrt{\alpha_c z \bar{z}}} J_1(2 \sqrt{\alpha_c z \bar{z}}) \quad (11)$$

where \bar{A}_0 is an arbitrary constant matrix.

The next task is to examine whether there are any possible solutions which are “non-abelian” in their nature. Consider in the special case of $SU(2)$ the configuration $\bar{A} = e^{-i\theta\tau_0}\bar{A}_0e^{i\theta\tau_0}$ with a constant matrix $\bar{A}_0 = e_0\tau_0 + \bar{e}\tau + e\bar{\tau}$. Plugging this ansatz into eqn. (8) with no external source one finds that there is a solution provided that $\partial\theta$ and $\bar{\partial}\theta$ are constants, namely $\theta = \theta_0 + k\bar{z} + \bar{k}z$ where k , \bar{k} and θ_0 are constants. Indeed the following gauge field

$$\bar{A} = \frac{\tilde{\alpha}_c - k\bar{k}}{\bar{k}}\tau_0 + \sqrt{\frac{(k\bar{k} - \tilde{\alpha}_c)\tilde{\alpha}_c}{4\bar{k}^2}}[e^{-i\theta}\tau + e^{i\theta}\bar{\tau}] \quad (12)$$

is a “non-abelian solution”. The notation and the derivation are presented in appendix A. Setting $\theta_0 = 0$ requires that $(k\bar{k} - \tilde{\alpha}_c) > 0$. Looking into the case where \bar{A}_0 is not a constant, but with $\partial\bar{A}_0 = 0$ to preserve $F(\bar{A}_0) = 0$, one gets that the only solution is with a constant \bar{A}_0

The solution we have found eqn. (12) are truly non-abelian solution. The corresponding F is $F = i\bar{k}e^{-i\theta\tau_0}(\bar{e}\tau - e\bar{\tau})e^{i\theta\tau_0}$. Performing a gauge transformation with $U = e^{-i\theta\tau_0}$ one finds that $F_U = i\bar{k}(\bar{e}\tau - e\bar{\tau})$, $\bar{A}_U = \bar{A}_0 + k\tau_0$ and $A_U = \bar{k}\tau_0$ (recall that $A = 0$). We thus found that A_U , \bar{A}_U and F_U are fixed in space-time and no two commuting. Furthermore an abelian gauge configuration of the form $\bar{A} = -i\bar{k}z(\bar{e}\tau - e\bar{\tau})$ and $A = 0$ that leads to the same F is not connect to A_U , \bar{A}_U by a gauge transformation.

Using the expression of \bar{A} one can easily extract j_{dy} and \bar{j}_{dy} . This will be done in section 6. Moreover, one can determine the non-abelian group factor h . From eqn.(6) it follows that $h^{-1}\partial h = i\frac{\partial^2\bar{A}}{\alpha_c}$. Using the ansatz for $\bar{A} = e^{-i\theta\tau_0}\bar{A}_0e^{i\theta\tau_0}$ it is easy to find that

$$h = e^{-i\bar{k}z[\tau_0 + \tilde{e}(\tau + \bar{\tau})]}e^{i\theta\tau_0} \quad (13)$$

where $\tilde{e} = \frac{1}{2}\sqrt{\frac{\bar{k}k}{\alpha_c} - 1}$.

5. SOLUTIONS OF THE EQUATIONS WITH EXTERNAL CURRENT

Next we want to turn on a covariantly conserved (eqn. (7)) external current J_{ext} and study the corresponding equations of motion. Abelian solutions are easily constructed. For instance for a pair of quark anti-quark as an external classical source $\bar{J}^a_{ext} = T^a \delta^{a,a_0} Q [\delta(x_1 - R) - \delta(x_1 + R)]$ the abelian solution is

$$\bar{A} = \frac{1}{2} \sqrt{2\alpha_c} T^a \delta^{a,a_0} Q [e^{-\sqrt{2\alpha_c}|x_1-R|} - e^{-\sqrt{2\alpha_c}|x_1+R|}]. \quad (14)$$

Inserting this expression into $\frac{1}{2\pi} \int dx_1 Tr[A \bar{J}_{ext}]$ one finds the usual screening potential^[2]

$$V(r) = \frac{1}{2\pi} \sqrt{2\alpha_c} Q^2 (1 - e^{-2\sqrt{2\alpha_c}|R|}) Tr[(T^{a_0})^2] \quad (15)$$

Again the challenge is to find “non-abelian” solutions where the commutator terms do not vanish. The $SU(2)$ “non-abelian solution” of above is a solution also in case of a constant external current $\bar{J} = \tau^a \delta^{a,0} J_0$ with the trivial modification that g (see appendix A) is replaced by $\sqrt{\frac{\tilde{\alpha}_c [\frac{J_0 \bar{k}}{N_f} + (k\bar{k} - \tilde{\alpha}_c)]}{4k^2}}$. Consider now an external current of the form $\bar{J} = \bar{J}_0(\bar{z})\tau_0$. A solution in that case is

$$\bar{A} = (f_0 + J_0(\bar{z}))\tau_0 + [g(e^{[-i(k\bar{z} + \bar{k}z + I)])}\bar{\tau} + c.c]$$

where $\bar{\partial}I(\bar{z}) = \frac{1}{N_f} \bar{J}_0(\bar{z})$ with f_0 and g related to k and \bar{k} as given in eqn. (12). In the case of light-front quantization with \bar{z} playing the role of the space coordinates, $\bar{J}_{ext}(\bar{z})$ stands for a general “static” current. In particular a current density that corresponds to a quark anti-quark pair takes in this framework the form $\bar{J}_{ext}^a = \frac{1}{2} \tau^a \delta^{a,a_0} Q [\delta(\bar{z} - R) - \delta(\bar{z} + R)]$ and the corresponding solution has $\epsilon(\bar{z} - R)$ and $\epsilon(\bar{z} + R)$ factors in θ . The corresponding potential is a constant thus non-confining.

6. THE ENERGY-MOMENTUM TENSOR AND THE SPECTRUM

Next we want to analyze the spectrum of physical states that correspond (at least in the large N_f limit) to solutions of the equations of motion. Recall that those states transform in the adjoint representation of the global color transformations.

First we have to compute the energy momentum tensor $T \equiv T_{zz}, \bar{T} \equiv T_{\bar{z}\bar{z}}, T_{\bar{z}z}$ that corresponds to the action (3). Only the colored part of the energy momentum tensor is relevant to our discussion. From appendix B

$$T = \frac{\pi}{N_f + N_c} : Tr[j_{dy} j_{dy}] : \quad T_{\bar{z}z} = \frac{1}{8\pi\alpha_c} Tr[(\partial\bar{A})^2] \quad (16)$$

where the currents of the dynamical quarks which were defined below eqn.(6) are

$$\begin{aligned} j_{dy} &= \frac{iN_f}{2\pi} h^{-1} \partial h = -\frac{1}{2\pi\alpha_c} \partial^2 \bar{A}; \\ \bar{j}_{dy} &= \frac{1}{2\pi\alpha_c} (\partial\bar{\partial}\bar{A} + i[\partial\bar{A}, \bar{A}]) = -\frac{iN_f}{2\pi} h^{-1} \bar{\partial} h = -\frac{N_f}{2\pi} \bar{A} \end{aligned} \quad (17)$$

To proceed and compute the masses of the physical states one has to choose a quantization scheme. It is natural in the light cone gauge to use a light front quantization. In that scheme we take z to denote the space coordinate. In appendix B we express the momentum components \bar{P} and P as integrals over T and $T_{\bar{z}z}$. The masses of the states are given by the eigenvalues of $M^2 = P\bar{P}$.

To set the proper normalization of the fields let us consider first the abelian solutions for \bar{A} . In that case the operator \bar{A} can be written as

$$\bar{A}_{ab} = T^I e_c \int \frac{dk}{\mathcal{N}(k)} [a(k) e^{-i(k\bar{z} + \bar{k}z)} + c.c.]$$

where T^I is a matrix in the Cartan sub-algebra, $k\bar{k} = \tilde{\alpha}_c$, the creation and annihilation operators obey the commutation relation $[a(k), a^\dagger(\tilde{k})] = \delta(k - \tilde{k})$ and $\mathcal{N}(k)$ is a normalization factor. Inserting this form of \bar{A} it is a straightforward calculation to get $\mathcal{N}(k) = 2\sqrt{\pi k}$ so that M^2 on the states $|k, \bar{k}\rangle$ is equal as expected to $\tilde{\alpha}_c$.

One can instead assume a finite system of size L in z direction. In that case if one uses a normalization where $\bar{k}L$ is an integer n times 2π , and that $\bar{A}_{ab} = \frac{2}{\sqrt{n}}k\sin\theta$ (in first quantized version). One then finds that $P = k$ and $\bar{P} = \bar{k}$, so that again $M^2 = \tilde{\alpha}_c$.

In case of the $SU(N_c = 2)$ non-abelian solutions the superposition principle does not apply and there is no room for Fourier expansion of fields. We thus invoke the second quantization method of above, with a finite size system. The expectation values of the energy momentum tensor in those states are determined by substituting the expression for the gauge configuration \bar{A} into eqn.(16). This leads to

$$\begin{aligned} \langle T \rangle &= \frac{1}{8\pi} \frac{N_f^2}{N_f + N_c} (\bar{k})^2 \frac{k\bar{k}}{\tilde{\alpha}_c} \left(1 - \frac{\tilde{\alpha}_c}{k\bar{k}}\right) \\ \langle T_{\bar{z}z} \rangle &= \frac{N_f}{16\pi} (k\bar{k}) \left(1 - \frac{\tilde{\alpha}_c}{k\bar{k}}\right) \end{aligned} \quad (18)$$

If we use the same quantization scheme as for the abelian case we get that

$$\bar{P} = \frac{1}{8\pi} \frac{N_f^2}{N_f + N_c} (L\bar{k})\bar{k} \frac{k\bar{k}}{\tilde{\alpha}_c} \left(1 - \frac{\tilde{\alpha}_c}{k\bar{k}}\right) \quad P = \frac{N_f}{16\pi} (L\bar{k})k \left(1 - \frac{\tilde{\alpha}_c}{k\bar{k}}\right)$$

where L is the size of the system. Taking again the normalization $\bar{k}L = 2\pi n$ one finds that the non-abelian state is characterized by masses

$$M^2 = \frac{n^2}{32} \frac{N_f^3}{N_f + N_c} (k\bar{k}) \frac{k\bar{k}}{\tilde{\alpha}_c} \left(1 - \frac{\tilde{\alpha}_c}{k\bar{k}}\right)^2$$

The discussion above was all for $\theta_0 = 0$. Thus we require that $(k\bar{k} - \tilde{\alpha}_c) > 0$ (the case of zero $k\bar{k} - \tilde{\alpha}_c$ corresponds to vanishing \bar{A}). We get an M starting from $M = 0$ and growing up linearly in $k\bar{k}$ for $k\bar{k} \gg \tilde{\alpha}_c$. Note also that our solution is singular for $\tilde{\alpha}_c = 0$.

7. BOSONIZED EXTERNAL CURRENTS

Another approach to the coupling of the dynamical fermions to external currents is to bosonize the “external” currents. Let us briefly summarize first the abelian case. Consider external fermions of mass M and charge qe described by the real scalar field Φ together with the dynamical fermions of unit charge e and mass m associated with the scalar ϕ . The Lagrangian of the combined system after integrating out the gauge fields is given by

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi) + m\Sigma[\cos(2\sqrt{\pi}\phi) - 1] + \frac{1}{2}(\partial_\mu \Phi \partial^\mu \Phi) + M\Sigma[\cos(2\sqrt{\pi}\Phi) - 1] \\ & - \frac{e^2}{2\pi}(\phi + q\Phi)^2 \end{aligned} \tag{19}$$

Let us look for static solutions of the corresponding equations of motion with finite energy. Take, without loss of generality, $\phi(-\infty) = \Phi(-\infty) = 0$. From the M term we get $\Phi(\infty) = \sqrt{\pi}N$ with N integer. For $m \neq 0$ we also get $\phi(\infty) = \sqrt{\pi}n$. Now from the e^2 term, $n + qN = 0$. Thus, for instance for $N = 1$ finite energy solutions occur only for $q = -n$. In the massless case we have only $\Phi(\infty) = \sqrt{\pi}N$ and then a finite energy solution for $N = 1$ is if $\phi(\infty) = -\sqrt{\pi}q$. So, when $q = -n$, the system is always in the screening phase, whereas when $q \neq -n$ it is in the confinement phase for $m \neq 0$ and screening for $m = 0$.

Proceeding now to the QCD case one can consider several different possibilities $(J_{ext}^F, j_{dy}^F), (J_{ext}^F, j_{dy}^{ad}), (J_{ext}^{ad}, j_{dy}^F), (J_{ext}^{ad}, j_{dy}^{ad})$ and with dynamical fermions that can be either massless or massive. Obviously, for the external source one would assign a mass which should then be taken to infinity. The system of dynamical adjoint fermions and external fundamental quarks can be described by an action which is the sum of (4) and (1).

Integrating over the gauge degrees of freedom one is left with the terms in (1) and (4) that do not include coupling to gauge field together with a current-current non-local interaction term. For the interesting case of dynamical quarks in the

adjoint and external in fundamental we get for the interaction term

$$\int d^2z \sum_a \left\{ \frac{1}{\partial} [Tr(T_F^a i u_{ext}^{-1} \partial u_{ext}) + \frac{i}{2} Tr(T_{ad}^a h^{-1} \partial h)] \right\}^2 \quad (20)$$

where u is defined in (1), T_F^a are the $SU(N_c)$ generators expressed as $(N_c N_f) \times (N_c N_f)$ matrices in the fundamental representation of $U(N_c \times N_f)$ and T_{ad}^a the $(N_c^2 - 1) \times (N_c^2 - 1)$ matrices in the $SU(N_c)$ adjoint representation. The other cases can be treated similarly. For simplicity we discuss from here on the case of a single flavor.

Let us first consider the case of external adjoint quarks $u_{ext}(x) \in SU(N_c) \times U_B(1)$, which can be represented as

$$u_{ext} = \begin{pmatrix} e^{-i\Phi} & & & \\ & e^{i\Phi} & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{pmatrix} \quad (21)$$

(Φ is not normalized canonically here). The reason that we take a diagonal ansatz is that it corresponds, as we argue in appendix C, to a minimal energy configuration. Ansatz (21) corresponds to $Q_{ext}^1 \bar{Q}_{ext}^2$, namely, to an external adjoint state. We expect this state to be screened by the adjoint dynamical fermions. With this ansatz $\frac{1}{\partial}(i h_{ext}^{-1} \partial h_{ext})$ takes the form

$$\begin{pmatrix} \Phi & & & \\ & -\Phi & & \\ & & 0 & \\ & & & \ddots \\ & & & & 0 \end{pmatrix} \quad (22)$$

It is thus clear that only T_F^3 contributes to the trace in (20). To show the dynamical

configuration that screens, take

$$\log h_{ad} = \begin{pmatrix} 0 & -\phi & 0 & & \\ \phi & 0 & 0 & & \\ 0 & 0 & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} \quad (23)$$

with the matrix that contributes to the Tr in (20),

$$T_{adj} = i \begin{pmatrix} 0 & 1 & 0 & & \\ -1 & 0 & 0 & & \\ 0 & 0 & 0 & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \quad (24)$$

which corresponds to the generator of rotation in direction 3 for the sub $O(3)$ of first three indices, thus obtaining the a term proportional to $(\Phi + \phi)^2$ emerging from (20). The mass terms for u_{ext} and h are now proportional to $(1 - \cos\Phi)$ and $(1 - \cos\phi)$ respectively. A boundary condition $\Phi(\infty) = 2\pi$ can be cancelled in the interaction term by the boundary condition $\phi(\infty) = -2\pi$.

Let us examine now the case of a single external quark

$$u_{ext} = \begin{pmatrix} e^{-i\Phi} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \quad (25)$$

Its contribution to the interaction term is $\sum_{i=1}^{N_c-1} (\eta_i \Phi + \text{"dyn"})^2$ where $\eta_i = \frac{1}{\sqrt{2i(i+1)}}$ and "dyn" is the part of the dynamical quarks. If again we take a configuration of the dynamical quarks based on a single scalar like in (23) we get altogether an e^2 term of the form $(\frac{1}{2}\Phi + \phi)^2$. Now if $\Phi(\infty) = 2\pi$ one cannot find a finite energy solution since from the mass term $\phi(\infty) = 2\pi n$, and thus there is no

way to cancel the interaction term. If, however, we consider massless dynamical fermions there is no constraint on $\phi(\infty)$ so it can be taken to be equal $-\pi$, and thus again a screening situation is achieved. This argument should be supplemented by showing that one cannot find another configuration besides (23) that may cancel the $\eta_i\Phi$ term in $(\eta_i\Phi + \text{“dyn”})^2$, for $SU(N_c)$ with $N_c \geq 3$.

8. DISCUSSION

In the present paper, using bosonization techniques, we have presented further evidence for the non-abelian screening of external charges by dynamical massless fermions. We have shown it explicitly for dynamical fermions in the adjoint and N_f fundamental representations. In fact, the latter case implies that a WZW model coupled to non-abelian gauge fields is a screening model for any level of the affine Lie algebra. The fact that there is no relation between the charges of the screening dynamical fermions and those of the external sources may seem unintuitive. However, one can understand this phenomena in a simple way if one realizes that the interaction between the external charges involves an exchange of a massive mode which is an outcome of the dynamics of bare massless quarks. The main outcome of the present paper is the observation that indeed such massive modes manifest themselves in the form of solutions of the equations of motion. This is well known for QED_2 , and has been emphasized more recently for QCD_2 .^[6]

A natural question to ask is whether there are consequences of the screening behaviour in the spectrum of the theory. A simple minded intuition of the difference between a confining and a screening spectrum can be derived from quantum mechanics. A potential of the form (15) leads to a spectrum of bound states with energy smaller than the asymptotic value of the potential. A linear potential, on the other hand, can accomodate an infinite spectrum of bound states with no limitations on their energies. Practically, of course, higher energy states will be unstable. It is obvious that the massless Schwinger model which has a single state in its spectrum falls into the former class. It looks plausible that the spectrum in the non-abelian case is also limited.

Another way to distinguish between confining and non-confining spectrum is the dependence of multiplicity of the physical states on the number of colors. In a confining spectrum one finds only color singlets and their multiplicities do not depend directly on N_c . The massive states discussed in the present work, both the abelian and the non-abelian solutions, admit a degeneracy of $N_c^2 - 1$, or stated differently those states are in the adjoint of the “global color symmetry”.

We have used an argument that in the large N_f limit the classical solutions of the equations of motion dominate the functional integral. However, it is not obvious that the corresponding states are physical. In a previous paper^[6] we have used a special formulation of QCD_2 in terms of $A = if^{-1}\partial f$, $\bar{A} = i\bar{f}\bar{\partial}\bar{f}^{-1}$ with $f(z, \bar{z}), \bar{f}(z, \bar{z}) \in [SU(N_c)]^c$ the complexification of $SU(N_c)$. Implementing BRST techniques we have solved for the physical states of the abelian analog. In the non-abelian case one finds massive color singlet states with a mass of $e_c\sqrt{\frac{N_f}{2\pi}}$ which are the analogs of the states discussed in the present paper. Unlike the massless partners of these states, for the massive ones we were not able to show that they are not physical states. We still do not have a definite answer to this question; however, the fact that in the abelian theory of the large N_f limit they are physical states supports the conjecture that they are physical also for finite N_f . Furthermore, it seems that no matter whether they are in the sub-space of physical states or not they are responsible for the screening potential.

One may suspect that the massive “Schwinger like” states are an artifact of the abelianization of the theory in the large N_f limit. To exclude this possibility we have found truly non-abelian solutions of the equations of motions.

The fact that an abelian nature is not necessary for the existence of the massive modes tells us that it is plausible that the spectrum is characterized by a smooth transition of the screening behaviour from large N_f down to $N_f = 1$. It may seem that there is a contradiction between the seminal work of ’t Hooft and the various types of evidence of the screening behavior of the massless theory. The reason for that is that the analysis of ref. [8], which predicts a confining spectrum, is

insensitive to the question of whether the quarks are massless or massive. The way to reconcile the two pictures of the massless model is the following. Assume that the potential is of the form of eq. (15). In the approach of [8], $e_c^2 N_c$ is kept finite in the large N_c limit. This implies that the potential behaves like $(1 - e^{-\frac{\kappa}{\sqrt{N_c}}|R|}) \sim \frac{\kappa}{\sqrt{N_c}}|R|[(1 - \frac{1}{2}\frac{\kappa}{\sqrt{N_c}}|R| + o(\frac{1}{N_c}))]$ for fixed R and large N_c , with $\kappa = 2\sqrt{2\alpha_c N_c}$ a finite constant. Now it is clear that in the limit of $N_c \rightarrow \infty$ the potential looks like a linear potential which obviously admits a confinement behaviour. Thus, the large N_c limit prevents one from detecting the truly screening nature of the massless system.

In the present paper we have not discussed the solutions of the equations of motion and the corresponding potentials for the case of massive quarks (for large N_f it is discussed in [12]). However, using the analogy with the massive Schwinger model one can get a general picture of the passage to a confining behavior. The mass of the massive state of the Schwinger model is shifted once quark mass is turned on. But an additional massless state emerges. Exchange of the latter mode causes confinement. Presumably a similar situation occurs in the non-abelian case. Moreover, in the double bosonization description of the massive model where the external sources are also bosonized, the confining nature of the theory manifest itself via the absence of quark soliton solutions. This fits well with the confining spectrum discovered by 't Hooft, the baryonic spectrum analyzed in ref. [7] and the results of ref. [2]. One can envisage having a term $m^2|R|$ in the potential, where m is the quark mass, in addition to the screening term, with only a screening remaining for $m = 0$.

Several open questions arise following the results of the present work, for instance the quantization of the non-abelian solutions of the equations of motion, deriving similar solutions for the massive model, etc. In the introduction the long standing puzzle of confinement in non-abelian gauge theories in 4D was mentioned as one of the motivations for the present work, so we cannot end this discussion without addressing the question of the relevance of our results to the real world QCD. The screening of external quarks in the fundamental representation

by dynamical fermions in the same representation both in the massless and massive cases seems to fit the picture one has about 4D QCD with fundamental quarks. The analysis of 2D model with adjoint fermions exhibit a screening behaviour for massless quarks and confinement one for massive case. In 4D one believes that adjoint quarks cannot screen external fundamental ones. The mass of the dynamical fermions does not play any role in this issue in 4D. This phenomenon that the nature of the mass term in 2D is very different than the one in 4D, was found also in other circumstances like the baryonic spectrum.^[7] One may speculate that the analog of quarks in 4D, whether massive or massless, are massive quarks in 2D. In [2] it is speculated that the analog of the phase transition that occurs by turning on mass in the adjoint case is that of breaking SUSY and losing screening in $N = 1$ supersymmetric YM in 4D.

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APPENDIX A

Non-abelian solution of the equation of motion for $SU(2)$

Let us parametrize the $SU(2)$ gauge field in the following form $\bar{A} = f\bar{\tau} + \bar{f}\tau + f_0\tau_0$ where f_0 is real, \bar{f} is the complex conjugate of f and the matrices τ obey the algebra $[\tau_0, \tau] = \tau$; $[\tau_0, \bar{\tau}] = -\bar{\tau}$; $[\tau, \bar{\tau}] = 2\tau_0$. In terms of those variable the equation of motion eqn (8) read (with $\bar{J}_{ext} = 0$)

$$\begin{aligned} (\partial\bar{\partial} + \tilde{\alpha}_c)f - i(f\partial f_0 - f_0\partial f) &= 0 \\ (\partial\bar{\partial} + \tilde{\alpha}_c)f_0 + 2i(\partial\bar{f}f - \bar{f}\partial f) &= 0 \end{aligned} \tag{A.1}$$

Using the notation $f = ge^{i\theta}$ eqn. (A.1) takes the form

$$\begin{aligned} (\partial\bar{\partial} + \tilde{\alpha}_c)g - g\partial\theta(f_0 + \bar{\partial}\theta) &= 0 \\ g\bar{\partial}\bar{\partial}\theta + \bar{\partial}\theta\partial g + \partial\theta\bar{\partial}g - (g\partial f_0 - f_0\partial g) &= 0 \\ (\partial\bar{\partial} + \tilde{\alpha}_c)f_0 + 4\partial\theta g^2 &= 0 \end{aligned} \tag{A.2}$$

The case where $\bar{\partial}\theta = \partial\theta = 0$ is the abelian solution since all the commutator terms vanish. In case that $\bar{\partial}\theta = k$; $\partial\theta = \bar{k}$ where k and \bar{k} are constants it is easy to check

that the equation do admit a non-abelian solution as eqn. (12) of the form

$$\theta = k\bar{z} + \bar{k}z; \quad f_0 = \frac{\tilde{\alpha}_c - k\bar{k}}{\bar{k}}; \quad g^2 = \frac{(k\bar{k} - \tilde{\alpha}_c)\tilde{\alpha}_c}{4\bar{k}^2}. \quad (\text{A.3})$$

This result follows also from the ansatz $\bar{A} = e^{-i\theta\tau_0}\bar{A}_0e^{i\theta\tau_0}$ (\bar{A}_0 constant) for which eqn. (8) takes the form

$$-\partial\theta\bar{\partial}\theta[\tau_0, [\tau_0, \bar{A}_0]] + i\partial\bar{\partial}\theta[\tau_0, \bar{A}_0] + \tilde{\alpha}_c\bar{A}_0 + \partial\theta[[\tau_0, \bar{A}_0], \bar{A}_0] = 0 \quad (\text{A.4})$$

inserting the values for the commutators one finds $\tilde{\alpha}_cf_0 + 4\bar{\partial}\theta g^2 = 0$, namely, $\bar{\partial}\theta = k$ where k is a constant. Using this property leads to a similar relation $\partial\theta = \bar{k}$ where \bar{k} is also a constant and to the determination of g and f_0 as given in eqn. (12). When an external source of the form $\bar{J}_{ext} = \tau_0\bar{J}_0$ is added the only modification of eqn. (A.2) is that the r.h.s of the last equation becomes $\alpha_c\bar{J}_0$. Similarly, the r.h.s. of (A.4) becomes $\alpha_c\bar{J}_{ext}$. When \bar{J}_0 is not a constant, but a function of \bar{z} , we need to add to θ a function of \bar{z} , as discussed in section 5.

APPENDIX B

Derivation of the Energy-momentum tensor of QED_2

In order to understand the energy-momentum tensor of QCD_2 , we start with the derivation in the abelian theory. The Lagrangian density of massless QED_2 is given in bosonized form by

$$\mathcal{L}_{QED} = \partial\phi\bar{\partial}\phi + \frac{1}{2}F^2 + \frac{e}{\sqrt{\pi}}[\partial\phi\bar{A} - \bar{\partial}\phi A] \quad (\text{B.1})$$

where $F = F_{z,\bar{z}} = \partial\bar{A} - \bar{\partial}A$. The corresponding equations of motion are

$$2\partial\bar{\partial}\phi = -\frac{e}{\sqrt{\pi}}F; \quad \partial F = \frac{e}{\sqrt{\pi}}\partial\phi; \quad \bar{\partial}F = \frac{e}{\sqrt{\pi}}\bar{\partial}\phi \quad (\text{B.2})$$

We have here the currents $j_{[U(1)]} = -\frac{1}{\sqrt{\pi}}\partial\phi$ and $\bar{j}_{[U(1)]} = \frac{1}{\sqrt{\pi}}\bar{\partial}\phi$. The definition of

the energy momentum is

$$\begin{aligned}
T_{\bar{z}z} &= \frac{\partial \mathcal{L}}{\partial(\partial\phi)}(\partial\phi) + \frac{\partial \mathcal{L}}{\partial(\partial\bar{A})}(\partial\bar{A}) - \mathcal{L} \\
&= \partial\phi\bar{\partial}\phi + F(\partial\bar{A}) + \frac{e}{\sqrt{\pi}}\partial\phi\bar{A} - \mathcal{L} \\
&= \frac{1}{2}F^2 + \bar{\partial}(FA)
\end{aligned} \tag{B.3}$$

In a similar way we have that

$$\begin{aligned}
T_{z\bar{z}} &= \frac{1}{2}F^2 - \partial(F\bar{A}) \\
T_{\bar{z}\bar{z}} &= \bar{\partial}\phi\bar{\partial}\phi + \frac{e}{\sqrt{\pi}}\bar{\partial}\phi\bar{A} \\
T_{zz} &= \partial\phi\partial\phi - \frac{e}{\sqrt{\pi}}\partial\phi A
\end{aligned} \tag{B.4}$$

Note that we have here the “canonical” energy-momentum, not the symmetric one, but it generates too the Poincare group.

In the gauge $A = 0$ the above expressions become

$$\begin{aligned}
T_{\bar{z}z} &= \frac{1}{2}F^2; \quad T_{z\bar{z}} = \frac{1}{2}F^2 - \partial(F\bar{A}) \\
T_{\bar{z}\bar{z}} &= \bar{\partial}\phi\bar{\partial}\phi + \frac{e}{\sqrt{\pi}}\bar{\partial}\phi\bar{A} \\
T_{zz} &= \partial\phi\partial\phi
\end{aligned} \tag{B.5}$$

We now take z as our “space” variable and \bar{z} as “time”. Thus

$$\begin{aligned}
P &= \int dz T_{zz} = \int dz (\partial\phi)^2 = \pi \int dz j_{[U(1)]}^2 \\
\bar{P} &= \int dz T_{z\bar{z}} = \int dz \frac{1}{2}F^2
\end{aligned} \tag{B.6}$$

In the non-abelian case, in the gauge $A = 0$, T_{zz} has no contribution from $u^{-1}\partial u\bar{A}$ term nor from the F^2 term. Thus it is the same as the one derived from

the ungauged WZW action, namely

$$T_{zz} = \frac{\pi}{N_f + N_c} : j^2 :$$

$T_{\bar{z}z}$ has no contribution from the WZW part of the action, and the contribution from $u^{-1}\partial u\bar{A}$ is cancelled by the same term in $-\mathcal{L}$. Thus there is a contribution only from F^2 term, ie.

$$T_{\bar{z}z} = \frac{1}{2}F^2$$

The corresponding light-cone component P is given by

$$P = \int dz T_{zz} = \int dz \frac{\pi}{N_f + N_c} : j^2 : .$$

For \bar{P} we need $T_{z\bar{z}}$ since $\bar{P} = \int dz T_{z\bar{z}}$, but as the difference with the symmetric energy-momentum tensor does not contribute to \bar{P} we also have $\bar{P} = \int dz T_{\bar{z}z} = \int dz \frac{1}{2}F^2$. Note from the explicit expression for the abelian case that indeed $T_{z\bar{z}}$ differs from $\frac{1}{2}F^2$ by a ∂ term that does not contribute to the integral defining \bar{P} .

APPENDIX C

Minimum of energy for soliton solutions

The expression of the Hamiltonian for static configurations of the (ungauged) massive WZW model is

$$H = \int d^2z Tr[\partial_i g \partial_i g^\dagger] + m^2 Tr[2 - (g + g^\dagger)] \quad (C.1)$$

This is also the massive non-linear σ model. Consider a diagonal solution g_0 of the equation of motion. A general non-diagonal configuration can be written as

$g = Bg_0B^\dagger$ where B is a $U(N)$ matrix that is time independent. Substitution of g into (C.1) one finds

$$H = \int d^2z Tr[(\partial_i g_0 + [B^\dagger \partial_i B, g_0])(\partial_i g_0^\dagger + [B^\dagger \partial_i B, g_0^\dagger]) + m^2 Tr[2 - (g_0 + g_0^\dagger)]] \quad (C.2)$$

The difference between the latter Hamiltonian and that which corresponds to g_0 is

$$H - H_0 = \int d^2z Tr \left\{ [B^\dagger \partial_i B, g_0][B^\dagger \partial_i B, g_0^\dagger] + 2B^\dagger \partial_i B (g_0^\dagger \partial_i g_0 + g_0 \partial_i g_0^\dagger) \right\} \quad (C.3)$$

For a diagonal solution g_0 the last expression reduces to

$$H - H_0 = \int d^2z Tr \left\{ [B^\dagger \partial_i B, g_0][B^\dagger \partial_i B, g_0^\dagger] \right\} > 0 \quad (C.4)$$

Thus if g is non-diagonal, it cannot be a classical solution, as after diagonalization to g_0 it will have a lower energy.